ON THE SECOND MOMENT ESTIMATE INVOLVING THE λ-PRIMITIVE ROOTS MODULO n

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Abstract. Artin's Conjecture on Primitive Roots states that a non-square non unit integer a is a primitive root modulo p for positive proportion of p. This conjecture remains open, but on average, there are many results due to P. J. Stephens (see [14], also [15]). There is a natural generalization of the conjecture for composite moduli. We can consider a as the primitive root modulo $(\mathbb{Z}/n\mathbb{Z})^*$ if a is an element of maximal exponent in the group. The behavior is more complex for composite moduli, and the corresponding average results are provided by S. Li and C. Pomerance (see [8], [9], and [10]), and recently by the author (see [6]). P. J. Stephens included the second moment results in his work, but for composite moduli, there were no such results previously. We prove that the corresponding second moment result in this case.

1. Introduction

[1] Let $a>1$ be an integer and n be a positive integer coprime to a. Denote by $\ell_a(n)$ the multiplicative order of a modulo n. Carmichael (see [2]) introduced the lambda function $\lambda(n)$ which is defined by the universal exponent of the group $(\mathbb{Z}/n\mathbb{Z})^*$. The lambda function can be evaluated by the following procedure:

$$\lambda(n) = \text{l.c.m.}(\lambda(p_1^{e_1}),\cdots,\lambda(p_r^{e_r})),$$

where

$$\begin{aligned}
&n = p_1^{e_1}\cdots p_r^{e_r} \quad (p_i\text{ 's distinct prime numbers}),\\
&\lambda(p^e) = p^{e-1}(p-1) \quad \text{if } p \text{ is odd prime, and } e\geq 1,\\
&\lambda(2^e) = 2^{e-1} \quad \text{if } 1\leq e\leq 2,\\
&\lambda(2^e) = 2^{e-2} \quad \text{if } e\geq 3, \text{ and}\\
&\lambda(1) = 1.
\end{aligned}$$

We say that a is a λ-primitive root modulo n if we have $\ell_a(n)=\lambda(n)$. Following the definitions and notations in [10],

$$R(n) = \#\{a\in(\mathbb{Z}/n\mathbb{Z})^* : a \text{ is a } \lambda\text{-primitive root modulo } n.\},$$

$$N_a(x) = \#\{n\leq x : a \text{ is a } \lambda\text{-primitive root modulo } n.\}.$$

When $n=p$ is a prime, $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic of order $p-1$ and $\lambda(p)=\phi(p)=p-1$. Then, λ-primitive roots are just called the primitive roots, and we have $R(p)=\phi(p-1)$. Define

$$P_a(x) = \#\{p\leq x : a \text{ is a primitive root modulo } p.\}.$$

Artin's Conjecture on Primitive Roots (AC) states that for non-square, non-unit integer a,

$$P_a(x)\sim A_a\mathrm{Li}(x). \tag{1}$$

Here, A_a is a positive constant depending on a that is a rational multiple of the Artin's constant

$$A=\prod_p\left(1-\frac{1}{p(p-1)}\right).$$

Assuming the Generalized Riemann Hypothesis (GRH), C. Hooley [2] proved (1). Unconditionally on average, P. J. Stephens [14] showed that:

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For $y > \exp(4\sqrt{\log x \log\log x})$, and for any $D > 0$,

$$\frac{1}{y}\sum_{a\le y} P_a(x) = A\mathrm{Li}(x) + O\left(x\log^{-D} x\right). \tag{2}$$

In [15], P. J. Stephens introduced a method to widen the range of y to $y > \exp(c\sqrt{\log x})$ for some absolute positive constant c without explicitly determining it. Then S. Kim [6] obtained an explicit $c = 3.42$.

For the composite moduli, S. Li [9] proved that for $y \ge \exp((\log x)^{3/4})$,

$$\frac{1}{y}\sum_{a\le y} N_a(x) \sim \sum_{n\le x}\frac{R(n)}{n}. \tag{3}$$

S. Li and C. Pomerance [10] improved the range of y to:

$$y \ge \exp((2+\epsilon)\sqrt{\log x \log\log x}), \tag{4}$$

for any positive ϵ. S. Kim [6] proved (3) with a wider range of y:

If $y > \exp(3.42\sqrt{\log x})$, then there exists a positive constant c_1 such that

$$\frac{1}{y}\sum_{a\le y} N_a(x) = \sum_{n\le x}\frac{R(n)}{n} + O\left(x\exp(-c_1\sqrt{\log x})\right). \tag{5}$$

In [6], the author speculated that the following would be true:

If $y > \exp(4.8365\sqrt{\log x})$, then

$$\frac{1}{y}\sum_{a\le y}\left(N_a(x) - \sum_{n\le x}\frac{R(n)}{n}\right)^2 \ll x^2\exp\left(-c_2\sqrt{\log x}\right) \tag{6}$$

for some absolute positive constant c_2. This speculation is based on the corresponding normal order result for prime moduli that can be deduced from [6, (4)]:

If $y > \exp(4.8365\sqrt{\log x})$, then

$$\frac{1}{y}\sum_{a\le y}\left(P_a(x) - \sum_{p\le x}\frac{R(p)}{p}\right)^2 \ll x^2\exp\left(-c_3\sqrt{\log x}\right) \tag{7}$$

for some absolute positive constant c_3. In this paper, it turns out that the speculated result for composite moduli is not true, even for a wider range of y (3.42 instead of 4.8365):

Theorem 1.1. *If $y > \exp(3.42\sqrt{\log x})$, then*

$$\frac{1}{y}\sum_{a\le y}\left(N_a(x) - \sum_{n\le x}\frac{R(n)}{n}\right)^2 \gg \frac{x^2}{(\log\log\log x)^2}. \tag{8}$$

By applying [13, Theorem 1.1], we are able to obtain the following refined version of [8, Theorem 2.3] which provides the explicit constant $\frac{6}{\pi^2 e^\gamma} \approx 0.341326$:

Theorem 1.2. *For large x, we have*

$$\sum_{n\le x}\frac{R(n)}{n} \ge \left(\frac{6}{\pi^2 e^\gamma} + o(1)\right)\frac{x}{\log\log\log x}. \tag{9}$$

Also, we are able to give an explicit constant in Theorem 1.1:

Theorem 1.3. *If $y > \exp(3.42\sqrt{\log x})$, then*

$$\frac{1}{y}\sum_{a\le y}\left(N_a(x) - \sum_{n\le x}\frac{R(n)}{n}\right)^2 \ge (C + o(1))\frac{x^2}{(\log\log\log x)^2} \tag{10}$$

where

$$C = \frac{36}{\pi^4 e^{2\gamma}}\left(\prod_p \left(1 + \frac{1}{p^5+p^4-p^3-p^2}\right) - 1\right) \approx 0.003692.$$

2. Preliminaries

Throughout the chapters 2 and 3, the parameter ϵ is allowed to be any positive constant, and c_1 is some absolute positive constant which is not necessarily the same at each occurrence. We adopt the notations and definitions in [10]: If we define

$$\Delta_q(n) = \#\{\text{cyclic factors } C_{q^v} \text{ in } (\mathbb{Z}/n\mathbb{Z})^* : q^v || \lambda(n)\},$$

then

$$R(n) = \phi(n) \prod_{q|\phi(n)} \left(1 - \frac{1}{q^{\Delta_q(n)}}\right). \tag{11}$$

Let $\mathrm{rad}(m)$ denote the square-free kernel of m. Let

$$E(n) = \{a \in (\mathbb{Z}/n\mathbb{Z})^* : a^{\frac{\lambda(n)}{\mathrm{rad}(n)}} \equiv 1 \pmod n\},$$

and we say that χ is an elementary character if χ is trivial on $E(n)$. For each square free $h|\phi(n)$, let $\rho_n(h)$ be the number of elementary characters mod n of order h. Then

$$\rho_n(h) = \prod_{q|h} (q^{\Delta_q(n)} - 1).$$

For a character χ mod n, let

$$c(\chi) = \frac{1}{\phi(n)} \sum_b{}' \chi(b) = \begin{cases} \frac{(-1)^{\mathrm{ord}(\chi)} R(n)}{\phi(n)\rho_n(\mathrm{ord}(\chi))} & \text{if } \chi \text{ is elementary,} \\ 0 & \text{otherwise.} \end{cases},$$

where the sum Σ' is over λ-primitive roots in $[1,n]$. Since $R(n) \le \phi(n)$,

$$|c(\chi)| \le \bar{c}(\chi),$$

where

$$\bar{c}(\chi) = \begin{cases} \frac{1}{\rho_n(\mathrm{ord}(\chi))} & \text{if } \chi \text{ is elementary,} \\ 0 & \text{otherwise.} \end{cases}$$

For the proof of above, see [10, Proposition 2].

Let $X(n)$ be the set of non-principal elementary characters mod n. In [8], the counting function of λ-primitive roots is defined and denoted by $t_a(n)$:

$$t_a(n) := \begin{cases} 1 & \text{if } a \text{ is a } \lambda\text{-primitive root modulo } n, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is possible to write $t_a(n)$ as a character sum:

$$t_a(n) = \sum_{\chi \bmod n} c(\chi)\chi(a).$$

Distinguishing the character by principal and non-principal, the sum $\sum_{a\le y} N_a(x)$ can be decomposed into three parts:

$$\sum_{a \le y} N_a(x) = y \sum_{n \le x} \frac{R(n)}{n} + B(x,y) + O(x \log x), \tag{12}$$

where

$$B(x,y) = \sum_{n\le x} \sum_{\chi \in X(n)} c(\chi) \sum_{a \le y} \chi(a). \tag{13}$$

Following the proof in [10],

$$|B(x,y)| \le \sum_{d\le x} |\mu(d)| S_d, \tag{14}$$

where

$$S_d = \sum_{\substack{k\le \frac{x}{d}\\ (k,d)=1}} \sum_{\substack{m_1\le \frac{x}{dk}\\ \mathrm{rad}(m_1)|k}} \sum_{\substack{m_2\le \frac{x}{dkm_1}\\ (m_2,k)=1}} \sideset{}{^*}\sum_{\chi \bmod k} |c(\chi\chi_{0,dkm_1m_2})| \left| \sum_{a\le \frac{y}{d}} \chi(a) \right|, \tag{15}$$

and Σ^* is over non-principal primitive characters.

Let $\chi_{0,n}$ be the principal character modulo n. For positive integer k and reals w,z, define

$$F(k,z) = \sum_{\mathrm{rad}(m)|k} \frac{1}{m} \sideset{}{^*}\sum_{\chi (\bmod\ k)} \bar{c}(\chi\chi_{0,km}) \left| \sum_{a\le z} \chi(a) \right|,$$

$$T(w,z) = \sum_{k\le w} F(k,z),$$

$$S(w,z) = w\sum_{k\le w} \frac{1}{k} F(k,z) = T(w,z) + w\int_1^w \frac{1}{u^2} T(u,z)du,$$

and

$$S_d \le S\left(\frac{x}{d}, \frac{y}{d}\right).$$

We exhibit here a series of lemmas in [6] that lead to the estimation of $B(x,y)$. The function f is defined by:

$$f(K) = \frac{1}{K}\left(\log\left(\frac{K^2}{2}+1\right)+1\right),$$

which is the same function as $f_1(K) + \frac{K}{4}$ as in [6].

Lemma 2.1. *(1) If* $z > \exp(4.18\sqrt{\log w})$*, then*

$$T(w,z) \ll wz^{\frac{13}{16}} \exp\left(\sqrt{\log w}\,(f(4.18)+\epsilon)\right). \tag{16}$$

(2) If $\exp(3.419906\sqrt{\log w}) < z \le \exp(16\sqrt{\log w})$*, then*

$$T(w,z) \ll wz^{\frac{3}{4}} \exp\left(\sqrt{\log w}\,(f(3.419906)+\epsilon)\right). \tag{17}$$

By $S(w,z) = T(w,z) + w\int_1^w \frac{1}{u^2} T(u,z)du$,

Lemma 2.2. *(1) If* $z > \exp(4.18\sqrt{\log w})$*, then*

$$S(w,z) \ll wz^{\frac{13}{16}} \exp\left(\sqrt{\log w}\,(f(4.18)+\epsilon)\right). \tag{18}$$

(2) If $\exp(3.419906\sqrt{\log w}) < z \le \exp(16\sqrt{\log w})$*, then*

$$S(w,z) \ll wz^{\frac{3}{4}} \exp\left(\sqrt{\log w}\,(f(3.419906)+\epsilon)\right) + w\log w \cdot z^{\frac{7}{8}}. \tag{19}$$

We also have by [11, Theorem 1] and partial summation,

Lemma 2.3.

$$S(w,z) \ll wz \exp\left(3\sqrt{\frac{\log w}{\log\log w}}\right).$$

Combining these, we obtain for $y > \exp(3.42\sqrt{\log x})$, there is a positive absolute constant c_1 such that

$$B(x,y) \ll xy\exp(-c_1\sqrt{\log x}). \tag{20}$$

3. Proof of Theorems

3.1. **Basic Set Up.** By (12), we have

$$\sum_{a\le y}\left(N_a(x)-\sum_{n\le x}\frac{R(n)}{n}\right)^2=\sum_{a\le y}\left(N_a(x)^2-2N_a(x)\sum_{n\le x}\frac{R(n)}{n}+\left(\sum_{n\le x}\frac{R(n)}{n}\right)^2\right)$$
$$=\sum_{a\le y}N_a(x)^2-y\left(\sum_{n\le x}\frac{R(n)}{n}\right)^2+O(x^2\log x)+O\left(xB(x,y)\right).$$

We treat the sum $\sum_{a\le y}N_a(x)^2$ first:

$$\sum_{a\le y}N_a(x)^2=\sum_{a\le y}\left(\sum_{n\le x}t_a(n)\right)^2=\sum_{a\le y}\sum_{n_1\le x}\sum_{n_2\le x}t_a(n_1)t_a(n_2)=\sum_{n_1\le x}\sum_{n_2\le x}\sum_{a\le y}t_a(n_1)t_a(n_2)$$
$$=\sum_{n_1\le x}\sum_{n_2\le x}\sum_{a\le y}\sum_{\chi_1 \bmod n_1}c(\chi_1)\chi_1(a)\sum_{\chi_2 \bmod n_2}c(\chi_2)\chi_2(a)$$
$$=y\sum_{n_1\le x}\sum_{n_2\le x}\frac{R(n_1)}{\phi(n_1)}\frac{R(n_2)}{\phi(n_2)}\frac{\phi(n_1n_2)}{n_1n_2}+\sum_{n_1\le x}\sum_{n_2\le x}\sum_{a\le y}\sideset{}{^\bullet}\sum_{\substack{\chi_1 \bmod n_1\\ \chi_2 \bmod n_2}}c(\chi_1)c(\chi_2)\chi_1\chi_2(a)+O(x^2\log^2 x)$$

where $\Sigma^\bullet$ is for at least one of χ_1 or χ_2 being non-principal. Then by (20),

$$\sum_{a\le y}\left(N_a(x)-\sum_{n\le x}\frac{R(n)}{n}\right)^2=y\sum_{n_1\le x}\sum_{n_2\le x}\frac{R(n_1)R(n_2)}{n_1n_2}\left(\frac{(n_1,n_2)}{\phi((n_1,n_2))}-1\right)$$
$$+\sum_{n_1\le x}\sum_{n_2\le x}\sum_{a\le y}\sideset{}{^\bullet}\sum_{\substack{\chi_1 \bmod n_1\\ \chi_2 \bmod n_2}}c(\chi_1)c(\chi_2)\chi_1\chi_2(a)$$
$$+O(x^2\log^2 x)+O\left(xB(x,y)\right)$$
$$=y\Sigma_1+\Sigma_2+E$$

where $E=O(x^2\log^2 x)+O\left(x^2y\exp(-c_1\sqrt{\log x})\right)$ if $y>\exp(3.42\sqrt{\log x})$.

3.2. **Estimation of $\mathbf{\Sigma_1}$.** Let $d=(n_1,n_2)$, $n_1=dk_1$, and $n_2=dk_2$, we may rewrite Σ_1 as

$$\Sigma_1=\sum_{n_1\le x}\sum_{n_2\le x}\frac{R(n_1)R(n_2)}{n_1n_2}\left(\frac{(n_1,n_2)}{\phi((n_1,n_2))}-1\right)$$
$$=\sum_{d\le x}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{R(dk_1)R(dk_2)}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)$$
$$\ge\sum_{d\le\sqrt{x}}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{\phi(\phi(dk_1))\phi(\phi(dk_2))}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)$$
$$\ge\sum_{d\le\sqrt{x}}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{\phi(\phi(k_1))\phi(\phi(k_2))}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)$$

As treated in [8, Theorem 2.3], we apply the normal order result for $\omega(\phi(n))$ in [3], and the existence of limiting density function of $\phi(n)/n$ in [16]. By [3], the set of numbers n which has at most $(\log\log n)^2$ distinct prime factors has asymptotic density 1. Let B_1 be the set of $k_1\le\frac{x}{d}$ such that $\phi(k_1)/k_1\ge 2/3$,

and B_2 be the set of $k_2 \le \frac{x}{d}$ such that $\phi(k_2)/k_2 \ge \epsilon_2 > 0$ which has asymptotic density $2/3$. Then there exists a set $\widetilde{B_2}$ of $k_2 \le \frac{x}{d}$ of asymptotic density at least $1/3$ satisfying the three conditions:

1. k_2 has at most $(\log\log k_2)^2$ distinct prime factors.
2. $(k_1, k_2) = 1$.
3. $\phi(k_2)/k_2 \ge \epsilon_2 > 0$.

Summing over $\widetilde{B_2}$, it follows that

$$\sum_{\substack{k_2 \le \frac{x}{d} \\ (k_1,k_2)=1}} \frac{\phi(\phi(k_2))}{k_2} \gg \frac{x}{d \log\log\log x}.$$

The remaining sum over k_1, we restrict the sum over $k_1 \in B_1$. Then we have a set $\widetilde{B_1}$ of $k_1 \le \frac{x}{d}$ of asymptotic density $\epsilon_1 > 0$ satisfying the conditions:

1. k_1 has at most $(\log\log k_1)^2$ distinct prime factors.
2. $\phi(k_1)/k_1 \ge 2/3$.

Summing over $\widetilde{B_1}$, it follows that

$$\sum_{k_1 \le \frac{x}{d}} \frac{\phi(\phi(k_1))}{k_1} \gg \frac{x}{d \log\log\log x}.$$

Thus, we obtain

$$\Sigma_1 \gg \frac{x^2}{(\log\log\log x)^2} \sum_{d \le \sqrt{x}} \frac{1}{d^2} \left(\frac{1}{d\phi(d)} - \frac{1}{d^2} \right) \gg \frac{x^2}{(\log\log\log x)^2}.$$

3.3. **Estimation of $\boldsymbol{\Sigma_2}$.** Recall that

$$\Sigma_2 = \sum_{n_1 \le x} \sum_{n_2 \le x} \sum_{a \le y} \sideset{}{^\bullet}\sum_{\substack{\chi_1 \bmod n_1 \\ \chi_2 \bmod n_2}} c(\chi_1) c(\chi_2) \chi_1\chi_2(a)$$

where $\Sigma^\bullet$ is for at least one of χ_1 or χ_2 is non-principal. Suppose that one of the characters, say $\chi_1 = \chi_{0,n_1}$ is principal. Then χ_2 is non-principal and induced by a non-principal primitive character. Thus, as in [10], the contribution in this case is

$$\le \sum_{n_1 \le x} \sum_{d \le x} |\mu(d)| \sum_{k \le \frac{x}{d}} \sum_{\substack{m_1 \le \frac{x}{dk} \\ \mathrm{rad}(m_1)|k}} \sum_{\substack{m_2 \le \frac{x}{dkm_1} \\ (m_2,k)=1}} \sideset{}{^*}\sum_{\chi_2 \bmod k} \overline{c}(\chi_2 \chi_{0,km_1}) \left| \sum_{a \le \frac{y}{d}} \chi_{0,n_1}\chi_2(a) \right|$$

$$\le \sum_{n_1 \le x} \sum_{d \le x} |\mu(d)| \sum_{k \le \frac{x}{d}} \sum_{\mathrm{rad}(m_1)|k} \frac{x}{dkm_1} \sideset{}{^*}\sum_{\chi_2 \bmod k} \overline{c}(\chi_2 \chi_{0,km_1}) \left| \sum_{a \le \frac{y}{d}} \chi_{0,n_1}\chi_2(a) \right|.$$

By Polya-Vinogradov inequality for imprimitive characters (see [12, p. 307]), Hölder inequality, and the large sieve inequality, we have

$$\sum_{k \le \frac{x}{d}} \sum_{\mathrm{rad}(m_1)|k} \frac{x}{dkm_1} \sideset{}{^*}\sum_{\chi_2 \bmod k} \overline{c}(\chi_2 \chi_{0,k_2m_2}) \left| \sum_{a \le \frac{y}{d}} \chi_{0,n_1}\chi_2(a) \right| \ll 2^{w(n_1)} S_2\left(\frac{x}{d}, \frac{y}{d}\right)$$

where $S_2(w, z)$ satisfies the same set of inequalities (Lemma 2.2 and 2.3) as $S(w, z)$. Here, we regard $\chi_1\chi_2$ as an imprimitive character induced by the primitive character χ_2. The contribution of this case is

$$\ll \sum_{n_1 \le x} 2^{w(n_1)} \sum_{d \le x} |\mu(d)| S_2\left(\frac{x}{d}, \frac{y}{d}\right)$$

Since S_2 satisfies Lemma 2.2, the contribution of $\frac{y}{d} > \exp\left(3.419906\sqrt{\log \frac{x}{d}}\right)$ is

$$\ll (x \log x) \sum_{d \le x} |\mu(d)| \frac{xy}{d^\alpha} \exp(-c_1\sqrt{\log x})$$

for some $\alpha > 1$ and $c_1 > 0$ if $y > \exp(3.42\sqrt{\log x})$. The contribution is $O(x^2 y \exp(-c_1\sqrt{\log x}))$. For $\frac{y}{d} \le \exp\left(3.419906\sqrt{\log \frac{x}{d_2}}\right)$, Lemma 2.3 for S_2 implies that the contribution is $O(x^2 y \exp(-c_1\sqrt{\log x}))$ if $y > \exp(3.42\sqrt{\log x})$.

Let χ_1 and χ_2 are both non-principal. As in [10], we begin with

$$\Sigma_2 = \sum_{\substack{n_1 \le x \\ n_2 \le x}} \sum_{\substack{k_1 | n_1 \\ k_2 | n_2}} \sideset{}{^{**}}\sum_{\substack{\chi_1 \bmod k_1 \\ \chi_2 \bmod k_2}} c(\chi_1\chi_{0,n_1}) c(\chi_2\chi_{0,n_2}) \sum_{a \le y} \chi_1\chi_2(a)\chi_{0,n_1}(a)\chi_{0,n_2}(a) \tag{21}$$

$$= \sum_{\substack{n_1 \le x \\ n_2 \le x}} \sum_{\substack{k_1 | n_1 \\ k_2 | n_2}} \sideset{}{^{**}}\sum_{\substack{\chi_1 \bmod k_1 \\ \chi_2 \bmod k_2}} c(\chi_1\chi_{0,n_1}) c(\chi_2\chi_{0,n_2}) \sum_{d | n_1 n_2} \chi_1\chi_2(d)\mu(d) \sum_{a \le \frac{y}{d}} \chi_1\chi_2(a) \tag{22}$$

where $\sum^{**}$ is for the pairs (χ_1, χ_2) with both χ_i are non-principal primitive. We use (21) for the case $\chi_1\chi_2$ is principal, use (22) otherwise.

Suppose that $\chi_1\chi_2$ is principal. In this case, $k_1 = k_2$ since χ_i are primitive, and $\chi_2 = \overline{\chi_1}$. Then we have

$$c(\chi_1\chi_{0,n_1}) = \begin{cases} \frac{(-1)^{\mathrm{ord}(\chi_1)} R(n_1)}{\phi(n_1)\rho_{n_1}(\mathrm{ord}(\chi_1))} & \text{if } \chi_1\chi_{0,n_1} \text{ is elementary modulo } n_1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c(\chi_2\chi_{0,n_2}) = c(\overline{\chi_1}\chi_{0,n_2}) = \begin{cases} \frac{(-1)^{\mathrm{ord}(\chi_1)} R(n_2)}{\phi(n_2)\rho_{n_2}(\mathrm{ord}(\chi_1))} & \text{if } \chi_2\chi_{0,n_2} \text{ is elementary modulo } n_2, \\ 0 & \text{otherwise.} \end{cases}$$

Since the terms $(-1)^{\mathrm{ord}(\chi_1)}$ cancel out, we see that $c(\chi_1\chi_{0,n_1})c(\overline{\chi_1}\chi_{0,n_2}) \ge 0$. Thus, the contribution of this case is

$$\sum_{\substack{n_1 \le x \\ n_2 \le x}} \sum_{\substack{k_1 | n_1 \\ k_1 | n_2}} \sideset{}{^{*}}\sum_{\chi_1 \bmod k_1} c(\chi_1\chi_{0,n_1}) c(\overline{\chi_1}\chi_{0,n_2}) \sum_{a \le y} \chi_{0,n_1 n_2}(a) \ge 0.$$

Suppose now that $\chi_1\chi_2$ is non-principal. Since $\mu(d) \ne 0$ only when d is square-free, we may use $d = d_1 d_2$ with $d_1 | n_1$ and $d_2 | n_2$. Moreover, $d \le x^2$ and we can impose $(d, k_1 k_2) = 1$ because of $\chi_1\chi_2(d)$. Thus, as in [10],

$$|\Sigma_2| \le \sum_{\substack{d \le x^2 \\ d = d_1 d_2}} |\mu(d)| \sum_{\substack{k_1 \le \frac{x}{d_1} \\ k_2 \le \frac{x}{d_2} \\ (d, k_1 k_2) = 1}} \sum_{\substack{m_1 \le \frac{x}{d_1 k_1} \\ \mathrm{rad}(m_1) | k_1 \\ m_2 \le \frac{x}{d_2 k_2} \\ \mathrm{rad}(m_2) | k_2}} \sum_{\substack{n_1 \le \frac{x}{d_1 k_1 m_1} \\ (n_1, k_1) = 1 \\ n_2 \le \frac{x}{d_2 k_2 m_2} \\ (n_2, k_2) = 1}} \sideset{}{^{**}}\sum_{\substack{\chi_1 \bmod k_1 \\ \chi_2 \bmod k_2}} |c(\chi_1\chi_{0,d_1 k_1 m_1 n_1}) c(\chi_2\chi_{0,d_2 k_2 m_2 n_2})| \left| \sum_{a \le \frac{y}{d}} \chi_1\chi_2(a) \right|.$$

By [10, Proposition 2, 3],

$$|\Sigma_2| \le \sum_{\substack{d \le x^2 \\ d = d_1 d_2}} |\mu(d)| \sum_{\substack{k_1 \le \frac{x}{d_1} \\ k_2 \le \frac{x}{d_2}}} \sum_{\substack{\mathrm{rad}(m_1) | k_1 \\ \mathrm{rad}(m_2) | k_2}} \frac{x^2}{d_1 d_2 k_1 k_2 m_1 m_2} \sideset{}{^{**}}\sum_{\substack{\chi_1 \bmod k_1 \\ \chi_2 \bmod k_2}} \overline{c}(\chi_1\chi_{0,k_1 m_1}) \overline{c}(\chi_2\chi_{0,k_2 m_2}) \left| \sum_{a \le \frac{y}{d}} \chi_1\chi_2(a) \right|.$$

Now, define

$$F(k_1, k_2, z) = \sum_{\substack{\mathrm{rad}(m_1) | k_1 \\ \mathrm{rad}(m_2) | k_2}} \frac{1}{m_1 m_2} \sideset{}{^{**}}\sum_{\substack{\chi_1 \bmod k_1 \\ \chi_2 \bmod k_2}} \overline{c}(\chi_1\chi_{0,k_1 m_1}) \overline{c}(\chi_2\chi_{0,k_2 m_2}) \left| \sum_{a \le z} \chi_1\chi_2(a) \right|,$$

$$T(w_1,w_2,z)=\sum_{\substack{k_1\le w_1\\k_2\le w_2}}F(k_1,k_2,z)=\sum_{k_1\le w_1}\sum_{\mathrm{rad}(m_1)|k_1}\frac{1}{m_1}\sideset{}{^*}\sum_{\chi_1 \mod k_1}\bar{c}(\chi_1\chi_{0,k_1m_1})T_2(\chi_1,w_2,z),$$

$$S(w_1,w_2,z)=w_1w_2\sum_{\substack{k_1\le w_1\\k_2\le w_2}}\frac{1}{k_1k_2}F(k_1,k_2,z)=w_1\sum_{k_1\le w_1}\frac{1}{k_1}S_2(k_1,w_2,z).$$

By partial summation applied two times, we have

$$\begin{aligned}S(w_1,w_2,z)&=T(w_1,w_2,z)+w_1\int_1^{w_1}\frac{1}{u_1^2}T(u_1,w_2,z)du_1+w_2\int_1^{w_2}\frac{1}{u_2^2}T(w_1,u_2,z)du_2\\&+w_1w_2\int_1^{w_1}\int_1^{w_2}\frac{1}{u_1^2u_2^2}T(u_1,u_2,z)du_2du_1.\end{aligned}$$

By Polya-Vinogradov inequality,

$$T(w_1,w_2,z)\ll w_1^{\frac{3}{2}}w_2^{\frac{3}{2}}\exp\left(3\sqrt{\frac{\log w_1}{\log\log w_1}}+3\sqrt{\frac{\log w_2}{\log\log w_2}}\right).\tag{23}$$

If $w=w_1w_2\le z^{\frac{3}{2}}$ then

$$T(w_1,w_2,z)\ll wz^{\frac{3}{4}}\exp\left(6\sqrt{\frac{\log w}{\log\log w}}\right).\tag{24}$$

If $w>z^{\frac{3}{2}}$ then one of w_1, or w_2 is greater than $z^{\frac{3}{4}}$, say $w_2>z^{\frac{3}{4}}$. By Hölder inequality, we have

$$T_2(\chi_1,w_2,z)^{2r}\le A^{2r-1}B,\tag{25}$$

where

$$A=\sum_{\substack{k_2\le w_2\\\mathrm{rad}(m_2)|k_2}}\frac{1}{m_2}\sideset{}{^*}\sum_{\chi_2(\mathrm{mod}\ k_2)}\bar{c}(\chi_2\chi_{0,k_2m_2})^{\frac{2r}{2r-1}},\tag{26}$$

and

$$B=\sum_{\substack{k_2\le w_2\\\mathrm{rad}(m_2)|k_2}}\frac{1}{m_2}\sideset{}{^*}\sum_{\chi_2(\mathrm{mod}\ k_2)}\left|\sum_{a\le z}\chi_1\chi_2(a)\right|^{2r}=\sum_{k_2\le w_2}\frac{k_2}{\phi(k_2)}\sideset{}{^*}\sum_{\chi_2(\mathrm{mod}\ k_2)}\left|\sum_{a\le z}\chi_1\chi_2(a)\right|^{2r}.\tag{27}$$

Then by $0\le\bar{c}(\chi_2\chi_{0,k_2m_2})\le1$,

$$A\ll w_2\exp\left(3\sqrt{\frac{\log w_2}{\log\log w_2}}\right).\tag{28}$$

By large sieve inequality,

$$B\ll(w_2^2+z^r)\sum_{a\le z^r}|\tau_r'(a)\chi_1(a)|^2\le(w_2^2+z^r)\sum_{a\le z^r}\tau_r'(a)^2.\tag{29}$$

If $w_2>z^{\frac{3}{4}}$, then $w_2^{\frac{1}{2r}}>z^{\frac{3}{20}}$. We apply the following lemma [6, Corollary 2.2]:

Lemma 3.1. *Let $c>0$. If $z\ge1$ and $r-1\le c\log z$, then*

$$\sum_{a\le z^r}(\tau_r'(a))^2\le\left(\frac{(1+c)^{r-1}}{(r-1)!}z\log^{r-1}z\right)^r,\tag{30}$$

with

$$z=\exp(K\sqrt{\log w_2}),\quad r=\left\lceil\frac{2\log w_2}{\log z}\right\rceil,\quad c=\frac{2}{K^2}.$$

By Stirling's formula, we obtain that

$$B^{\frac{1}{2r}}\ll z\exp\left(\sqrt{\log w_2}\,(f(K)+\epsilon)\right). \tag{31}$$

Therefore,

$$T_2(\chi_1,w_2,z)\ll w_2^{1-\frac{1}{2r}}z\exp\left(\sqrt{\log w_2}\,(f(K)+\epsilon)\right)\ll w_2z^{\frac{17}{20}}\exp\left(\sqrt{\log w_2}\,(f(K)+\epsilon)\right). \tag{32}$$

This yields

Lemma 3.2. *(1) If* $\exp\left(4.87\sqrt{\log w_2}\right)<z<w_2^{\frac{4}{3}}$, *then*

$$T_2(\chi_1,w_2,z)\ll w_2z^{\frac{17}{20}}\exp\left(\sqrt{\log w_2}\,(f(4.87)+\epsilon)\right). \tag{33}$$

(2) If $\exp\left(3.419906\sqrt{\log w_2}\right)<z\le\exp\left(20\sqrt{\log w_2}\right)$, *then*

$$T_2(\chi_1,w_2,z)\ll w_2z^{\frac{3}{4}}\exp\left(\sqrt{\log w_2}\,(f(3.419906)+\epsilon)\right). \tag{34}$$

By [11, Theorem 1],

Lemma 3.3. *(1) If* $z>\exp\left(4.87\sqrt{\log w_2}\right)$, *then*

$$T(w_1,w_2,z)\ll wz^{\frac{17}{20}}\exp\left(\sqrt{\log w_2}\,(f(4.87)+\epsilon)+3\sqrt{\frac{\log w_1}{\log\log w_1}}\right). \tag{35}$$

(2) If $\exp\left(3.419906\sqrt{\log w_2}\right)<z\le\exp\left(20\sqrt{\log w_2}\right)$, *then*

$$T(w_1,w_2,z)\ll wz^{\frac{3}{4}}\exp\left(\sqrt{\log w_2}\,(f(3.419906)+\epsilon)+3\sqrt{\frac{\log w_1}{\log\log w_1}}\right). \tag{36}$$

Finally, we have the estimate for $S(w_1,w_2,z)$:

Lemma 3.4. *(1) If* $z>\exp\left(4.87\sqrt{\log w_2}\right)$, *then*

$$S(w_1,w_2,z)\ll wz^{\frac{17}{20}}\exp\left(\sqrt{\log w_2}\,(f(4.87)+\epsilon)+3\sqrt{\frac{\log w_1}{\log\log w_1}}\right). \tag{37}$$

(2) If $\exp\left(3.419906\sqrt{\log w_2}\right)<z\le\exp\left(20\sqrt{\log w_2}\right)$, *then*

$$S(w_1,w_2,z)\ll wz^{\frac{3}{4}}\exp\left(\sqrt{\log w_2}\,(f(3.419906)+\epsilon)+3\sqrt{\frac{\log w_1}{\log\log w_1}}\right)+w(\log w)^2\exp\left(3\sqrt{\frac{\log w_1}{\log\log w_1}}\right)z^{\frac{9}{10}}. \tag{38}$$

By [11, Theorem 1] and partial summation,

Lemma 3.5.

$$S(w_1,w_2,z)\ll wz\exp\left(6\sqrt{\frac{\log w}{\log\log w}}\right). \tag{39}$$

Thus, the contribution of non-principal primitive χ_i's with $\chi_1\chi_2$ being non-principal, is

$$\ll\sum_{\substack{d\le x^2\\ d=d_1d_2}}|\mu(d)|S\left(\frac{x}{d_1},\frac{x}{d_2},\frac{y}{d}\right).$$

By Lemma 3.4, the contribution of $\frac{y}{d} > \exp\left(3.419906\sqrt{\log\frac{x}{d_2}}\right)$ is

$$\ll \exp\left(6\sqrt{\frac{\log x}{\log\log x}}\right)\sum_{d\le x^2}|\mu(d)|\tau(d)\frac{x^2y}{d^\alpha}\exp(-c_1\sqrt{\log x})$$

for some $\alpha > 1$ and $c_1 > 0$ if $y > \exp(3.42\sqrt{\log x})$. The contribution is $O\left(x^2y\exp\left(-c_1\sqrt{\log x}\right)\right)$. By Lemma 3.5, the contribution of $\frac{y}{d} \le \exp\left(3.419906\sqrt{\log\frac{x}{d_2}}\right)$ is also $O\left(x^2y\exp\left(-c_1\sqrt{\log x}\right)\right)$ provided that $y > \exp(3.42\sqrt{\log x})$. By the estimate of Σ_1, Theorem 1.1 follows.

3.4. **Proof of Theorem 1.2.** This is an easy consequence of [13, Theorem 1.1(i)] and partial summation. In fact, [13, Theorem 1.1(i)] states that:

Theorem 3.1 ((P. Pollack))**.**

$$\sum_{n\le x}\phi(\phi(n)) \sim \frac{3}{\pi^2e^\gamma}\frac{x^2}{\log\log\log x}.$$

Note that $R(n) \ge \phi(\phi(n))$. Thus,

$$\sum_{n\le x}\frac{R(n)}{n} \ge \sum_{n\le x}\frac{\phi(\phi(n))}{n}.$$

Let $A(t) = \sum_{n\le t}\phi(\phi(n))$. Then by partial summation,

$$\sum_{n\le x}\frac{\phi(\phi(n))}{n} = \frac{1}{x}A(x) + \int_{1-}^{x}\frac{A(t)}{t^2}dt \sim \frac{6}{\pi^2e^\gamma}\frac{x}{\log\log\log x}.$$

This completes the proof of Theorem 1.2. Indeed, this provides an alternative proof of [8, Theorem 2.3] with explicit constant $\frac{6}{\pi^2e^\gamma} \approx 0.341326$. Theorem 3.2 and its proof by P. Pollack will be greatly helpful toward the proof of Theorem 1.3.

3.5. **Proof of Theorem 1.3.** By Theorem 1.1, it is enough to consider Σ_1:

$$\Sigma_1 = \sum_{n_1\le x}\sum_{n_2\le x}\frac{R(n_1)R(n_2)}{n_1n_2}\left(\frac{(n_1,n_2)}{\phi((n_1,n_2))}-1\right) = \sum_{d\le x}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{R(dk_1)R(dk_2)}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right). \tag{40}$$

Since $R(n_i) \le x$, we have

$$\sum_{\sqrt{x}\le d\le x}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{R(dk_1)R(dk_2)}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right) \ll \sum_{\sqrt{x}\le d\le x}(x\log x)^2\frac{1}{d\phi(d)} \ll x^{\frac{3}{2}}(\log x)^2.$$

Thus, we may consider only

$$\sum_{d\le\sqrt{x}}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{R(dk_1)R(dk_2)}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right) \ge \sum_{d\le\sqrt{x}}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{\phi(\phi(dk_1))\phi(\phi(dk_2))}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right).$$

Then

$$\begin{aligned}
&\sum_{d\le\sqrt{x}}\sum_{k_1\le\frac{x}{d}}\sum_{\substack{k_2\le\frac{x}{d}\\(k_1,k_2)=1}}\frac{\phi(\phi(dk_1))\phi(\phi(dk_2))}{k_1k_2}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)\\
&=\sum_{d\le\sqrt{x}}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)\sum_{k\le\frac{x}{d}}\frac{\mu(k)}{k^2}\sum_{k_1\le\frac{x}{dk}}\sum_{k_2\le\frac{x}{dk}}\frac{\phi(\phi(dkk_1))\phi(\phi(dkk_2))}{k_1k_2}\\
&=\sum_{d\le\sqrt{x}}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)\sum_{k\le\frac{x}{d}}\frac{\mu(k)}{k^2}\left(\sum_{j\le\frac{x}{dk}}\frac{\phi(\phi(dkj))}{j}\right)^2\\
&=\sum_{d\le\sqrt{x}}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)\sum_{k\le x^{\frac14}}\frac{\mu(k)}{k^2}\left(\sum_{j\le\frac{x}{dk}}\frac{\phi(\phi(dkj))}{j}\right)^2+O(x^{\frac74}(\log x)^2).
\end{aligned}$$

Let A be the set of $n\le x$ of asymptotic density 1 with the property:

$$\frac{\phi(n)}{\phi(\phi(n))}\sim e^{\gamma}\log\log\log n,\qquad \text{along } n\to\infty \text{ in } A. \tag{41}$$

Let B be the complement of A in $\mathbb{Z}\cap[1,x]$. In the proof of Theorem 3.2, P. Pollack proved that

$$\sum_{n\le x,\ n\in B}\phi(\phi(n))=o\left(\frac{x^2}{\log\log\log x}\right). \tag{42}$$

It follows that uniformly for $1\le d\le x^{\frac34}$,

Corollary 3.1.

$$\sum_{n\le\frac{x}{d}}\phi(\phi(dn))=\sum_{n\le\frac{x}{d}}\frac{\phi(dn)}{e^{\gamma}\log\log\log dn}+o\left(\frac{x^2}{\log\log\log x}\right)=\frac{3x^2}{d\pi^2e^{\gamma}\log\log\log x}\prod_{p\mid d}\frac{1}{1+\frac1p}+o\left(\frac{x^2}{\log\log\log x}\right).$$

Therefore, we now have

$$\begin{aligned}
\Sigma_1&\ge\sum_{d\le\sqrt{x}}\left(\frac{1}{d\phi(d)}-\frac{1}{d^2}\right)\sum_{k\le x^{\frac14}}\frac{\mu(k)}{k^2}\frac{36x^2}{d^2k^2\pi^4e^{2\gamma}(\log\log\log x)^2}\prod_{p\mid dk}\frac{1}{\left(1+\frac1p\right)^2}+o\left(\frac{x^2}{(\log\log\log x)^2}\right)\\
&=(C+o(1))\frac{x^2}{(\log\log\log x)^2}
\end{aligned}$$

where

$$C=\frac{36}{\pi^4e^{2\gamma}}\left(\prod_p\left(1+\left(1+\frac1p\right)^{-2}\frac{1}{p^5}\left(1-\frac1p\right)^{-1}\right)-1\right)=\frac{36}{\pi^4e^{2\gamma}}\left(\prod_p\left(1+\frac{1}{p^5+p^4-p^3-p^2}\right)-1\right).$$

This completes the proof of Theorem 1.3.

4. Remarks

1. We observed that there is a significant difference in the problems involving prime moduli and composite moduli. This difference becomes evident in the normal order results which is mainly due to the following: For prime moduli, $(p,q)=1$ if p and q are distinct primes, but for composite moduli, (m,n) is not necessary equal to 1 for distinct m and n.

2. The classical large sieve inequality (see [5, Theorem 7.13]) is as follows:

$$\sum_{q\le Q}\frac{q}{\phi(q)}\sum_{\chi \bmod q}^{*}\left|\sum_{n\le N}a_n\chi(n)\right|^2\le(N+Q^2)\sum_{n\le N}|a_n|^2.$$

The key to bringing down the constant 4.8365 to 3.42 in [6], is by applying the large sieve inequality of the following form:

For any Dirichlet character χ_1, we have

$$\sum_{q\leq Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi \bmod q}\left|\sum_{n\leq N}a_n\chi\chi_1(n)\right|^2\leq (N+Q^2)\sum_{n\leq N}|a_n\chi_1(n)|^2\leq (N+Q^2)\sum_{n\leq N}|a_n|^2.$$

This allows us to treat the character double sum one by one rather than combining two characters into one as done in [14], [15], and [6]. Consequently, the constant 4.8365 in [6] can be now improved to 3.42. We summarize those improvement of results in [6]:

Theorem 4.1. *If $y>\exp(3.42\sqrt{\log x})$, then for any $D,E>0$, the following statements hold:*

$$\frac{1}{y}\sum_{a\leq y}P_a(x)=A\pi(x)+O\left(\frac{x}{\log^D x}\right), \tag{43}$$

$$\frac{1}{y}\sum_{a\leq y}(P_a(x)-A\pi(x))^2\ll\frac{x^2}{\log^E x}, \tag{44}$$

where $A=\prod_p\left(1-\frac{1}{p(p-1)}\right)$ is the Artin's constant, and

$$\frac{1}{y}\sum_{a\leq y}\sum_{p\leq x}\frac{\ell_a(p)}{p-1}=C\mathrm{Li}(x)+O\left(\frac{x}{\log^D x}\right), \tag{45}$$

$$\frac{1}{y}\sum_{a\leq y}\left(\sum_{p<x}\frac{\ell_a(p)}{p-1}-C\mathrm{Li}(x)\right)^2\ll\frac{x^2}{\log^E x}, \tag{46}$$

$$\frac{1}{y^2}\sum_{a\leq y}\sum_{b\leq y}\sum_{\substack{p\leq x\\ p\mid a^n-b\\ \text{for some } n}}1=C\mathrm{Li}(x)+O\left(\frac{x}{\log^D x}\right), \tag{47}$$

$$\frac{1}{y^2}\sum_{a\leq y}\sum_{b\leq y}\left(\sum_{\substack{p\leq x\\ p\mid a^n-b\\ \text{for some } n}}1-C\mathrm{Li}(x)\right)^2\ll\frac{x^2}{\log^E x}, \tag{48}$$

where $C=\prod_p\left(1-\frac{p}{p^3-1}\right)$ is the Stephens' constant.

3. Recall the function f defined by:

$$f(K)=\frac{1}{K}\left(\log\left(\frac{K^2}{2}+1\right)+1\right).$$

As in [6], the number 3.42 in Theorem 1.1 and 1.3, also 4.1 can be replaced by any number greater than ρ_1 where $\rho_1\approx 3.4199057$ is the unique positive root of the equation $\frac{K}{4}=f(K)$.

5. Further Developments

In [7], the author proved the following:

There exists $\delta>0$ such that if $x^{1-\delta}=o(y)$, then

$$\frac{1}{y}\sum_{a<y}\sum_{\substack{a<n<x\\ (a,n)=1}}\ell_a(n)=\frac{x^2}{\log x}\exp\left(B\frac{\log\log x}{\log\log\log x}(1+o(1))\right) \tag{49}$$

where

$$B = e^{-\gamma}\prod_p \left(1 - \frac{1}{(p-1)^2(p+1)}\right).$$

In a subsequent paper, the author will provide the above formula for $y > \exp(3.42\sqrt{\log x})$.

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